Journal of Statistical Physics, Vol. 100, Nos. 1/2, 2000

Charge Fluctuations in the Two-Dimensional One-Component Plasma

D. Levesque,[1] J.-J. Weis,[1] and J. L. Lebowitz[2]

Received August 10, 1999; final October 12, 1999

We study, via computer simulations, the fluctuations in the net electric charge in a two-dimensional, one component plasma (OCP) with uniform background charge density $-e\rho$ in a region Λ inside a much larger overall neutral system. Setting $e=1$, this is the same as the fluctuations in N_Λ, the number of mobile particles of charge e. As expected, the distribution of N_Λ has, for large Λ, a Gaussian form with a variance which grows only as $\hat{\kappa}\,|\partial\Lambda|$, where $|\partial\Lambda|$ is the length of the perimeter of Λ. The properties of this system depend only on the coupling parameter $\Gamma=kT$, which is the same as the reciprocal temperature in our units. Our simulations show that when the coupling parameter Γ increases, $\hat{\kappa}(\Gamma)$ decreases to an asymptotic value $\hat{\kappa}(\infty)\sim\hat{\kappa}(2)/2$ which is equal (or very close) to that obtained for the corresponding variance of particles on a rigid triangular lattice. Thus, for large Γ, the characteristic length $\xi_L=2\hat{\kappa}/\rho$ associated with charge fluctuations behaves very differently from that of the Debye length, $\xi_D\sim 1/\sqrt{\Gamma}$, which it approaches as $\Gamma\to 0$. The pair correlation function of the OCP is also studied.

KEY WORDS: Charge fluctuations; one-component plasma; two dimensions; perimeter law.

I. INTRODUCTION

A striking manifestation of the special long range nature of the Coulomb interactions and the resulting screening[1, 2] is that the fluctuations of the net electrical charge Q_Λ, contained in a subregion Λ of a spatially homogeneous and overall neutral equilibrium macroscopic system, grow only as the surface area $|\partial\Lambda|$ and not as its volume $|\Lambda|$, the normal

Dedicated to our friend and collegue George Stell.

[1] Laboratoire de Physique Théorique, Université de Paris-Sud, 91405 Orsay Cedex, France.
[2] Department of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903.

0022-4715/00/0700-0209$18.00/0

behavior of fluctuations of extensive variables (at critical points the growth is even faster).[3–8] This behavior of the charge fluctuations can be readily understood by considering the (truncated) charge-charge correlation function S in an overall neutral translation invariant Coulomb system. This is given by,

$$S(\mathbf{r}-\mathbf{r}') = \sum_{\alpha,\gamma} e_\alpha e_\gamma \langle \rho_\alpha(\mathbf{r}\, \rho_\gamma(\mathbf{r}')) \rangle \tag{1}$$

where $\rho_\alpha(\mathbf{r})$ is the microscopic density of particles of species α and e_α their charge. The charge fluctuations of Q_Λ is then expressed in terms of S as

$$\langle Q_\Lambda^2 \rangle = \int_\Lambda d\mathbf{r} \int_\Lambda d\mathbf{r}' S(\mathbf{r}-\mathbf{r}') \tag{2}$$

By introducing the characteristic function $\chi_\Lambda(\mathbf{r})$ of the domain Λ,

$$\chi_\Lambda(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \Lambda \\ 0, & \mathbf{r} \notin \Lambda \end{cases} \tag{3}$$

the integration in (2) can be extended to all space

$$\langle Q_\Lambda^2 \rangle = |\Lambda| \int_{\mathbb{R}^d} d\mathbf{r}\, S(\mathbf{r}) - \int_{\mathbb{R}^d} d\mathbf{r}\, S(\mathbf{r})\, \alpha_\Lambda(\mathbf{r}) \tag{4}$$

where

$$\alpha_\Lambda(\mathbf{r}) = \int_{\mathbb{R}^d} d\mathbf{r}'\, \chi_\Lambda(\mathbf{r}+\mathbf{r}')[1-\chi_\Lambda(\mathbf{r}')] \tag{5}$$

One observes now that the first term in (4), which is proportional to the volume, vanishes when the integral of S vanishes, i.e., if there is "perfect screening" of the charges.[3–7] Under these conditions the only contribution to (4) comes from the second term which yields, when the limit $|\Lambda| \to \mathbb{R}^d$ is taken in a self-similar way,

$$\lim_{|\Lambda| \to \infty} \frac{\langle Q_\Lambda^2 \rangle}{|\partial\Lambda|} = -\alpha_d \omega_d \int_0^\infty r^d S(r)\, dr = \left(\frac{1}{2} \sum_\alpha \rho_\alpha e_\alpha^2 \right) \xi_L \tag{6}$$

where $r = |\mathbf{r}|$, ω_d is the surface area of a unit sphere in d-dimensions ($\omega_1 = 2$), and it has been assumed that the system is also rotationally invariant. The geometrical constant α_d is: $\alpha_3 = 1/4$, $\alpha_2 = \pi^{-1}$, $\alpha_1 = 1/4$.[3] The constant ξ_L defines a characteristic length for the "charge separation" associated with the charge fluctuations.[3–8]

We thus see that perfect screening and existence of the integral in (6) imply surface growth of the variance of charge fluctuations. The latter, but not the former, will generally be violated when a charged system with d-dimensional Coulomb interactions is confined to a lower dimensional space, in which case the fluctuations will grow as $|\partial\Lambda| \log |\Lambda|$, e.g., for an r^{-1} potential in $d=2$ or logarithmic Coulomb potential in $d=1$.[9, 10] We also note that starting with a system which is not translation or rotation invariant, e.g., one with periodic structure, we can still apply (4) and, when perfect screening holds, also (6) after averaging over translations and rotations.

There are various arguments for expecting perfect screening, also referred to as the zeroth or charge "sum rule", in equilibrium Coulomb systems.[2–7] They all involve assumptions about some minimal decay of correlations in such systems. This has been proven rigorously for classical systems at sufficiently high temperatures and low densities, i.e., in the Debye–Hückel regime, where the decay is exponential.[11, 12] For quantum Coulomb systems the decay is only polynomial but still good enough. In fact one expects that perfect screening will always hold and so $\langle Q_\Lambda^2 \rangle / |\Lambda| \to 0$ and $|\Lambda| \to \infty$. Of course in order to treat particles with charges of different signs via classical statistical mechanics it is necessary to modify their Coulomb potential at short range, e.g., by introducing hard cores to prevent collapse. This is so in dimensions $d \geqslant 3$ at all temperatures and in $d=2$ at low temperatures. There is no such requirement for quantum systems as long as either the negative or positive charges (or both) obey Fermi statistics.[13]

II. PARTICLE FLUCTUATIONS IN THE OCP

A particularly interesting example of a system with reduced charge fluctuations is the one component plasma (OCP). In this system, used to model diverse physical situations, one kind of charges, say the negative ones, are treated as a uniform background with charge density $-\rho$, in which particles with positive unit point charges and average particle density ρ move about.[14, 15] Since the OCP background is fixed, charge fluctuations correspond to fluctuations in particle number, N_Λ. $S(\mathbf{r})$ in (4) now corresponds to the truncated particle-particle correlation function of a system with average particle density ρ i.e., $S(\mathbf{r}) = \{\rho\delta(\mathbf{r}) + \rho^2[\, g(\mathbf{r}) - 1]\}$ in the conventional liquid theory notation.

Surface area growth of the variance of particle fluctuations is a conceptually intriguing situation of interest beyond that of equilibrium Coulomb systems or even beyond statistical mechanics. Thus, it was

proven in ref. 16 that for any system of point particles the variance of N_Λ, averaged over translations and rotations, grows *at least* as fast as $|\partial\Lambda|$.[16] Examples of systems having such a variance are particles arranged on a periodic lattice structure or those obtained via small distortions of such structures.[16, 17] It was actually proven recently that in one dimension such growth (which in $d=1$ corresponds to bounded variance) implies, *by itself*, the existence of a periodic component in the extremal decomposition of any translation invariant measure, a fact already known for the $d=1$ OCP, using directly methods of equilibrium statistical mechanics.[18] The existence of such a periodic component implies that there cannot be good decay of all correlations in a translation invariant state. In fact the only example (known to us) of fluctuations in a point particle system with good mixing (decay of correlation) properties in $d>1$ is the OCP at high temperatures.[12] In $d=2$ the OCP is exactly solvable when the coupling parameter $\Gamma \equiv (kT)^{-1} = 2$.[19] The truncated correlations between groups of particles separated by a distance D decay in this system in a super-exponential way, like $\exp[-cD^2]$, with c computed explicitly. (Interestingly the distribution of particles in the OCP at $\Gamma=2$ is the same as that of the (suitably scaled) limiting distribution of eigenvalues of a random matrix M with entries $M_{ij} \equiv R_{ij} + iI_{ij}$ in which both R_{ij} and I_{ij} are independent identically distributed Gaussian random variables.[10])

In this note we study numerically the dependence of $\hat{\kappa}$ on the coupling constant Γ, for a two-dimensional OCP. Using a unit of length proportional to $\rho^{-1/2}$ the characteristic length $\xi_L = 2\hat{\kappa}/\rho$ depends only on Γ. We expect on physical grounds that $\xi_L(\Gamma)$ will decrease with Γ so the question is: how small can the fluctuations become when $\Gamma \to \infty$? On the one hand it is known from numerical studies that the 2d OCP undergoes some kind of ordering transition to a triangular lattice at $\Gamma = \Gamma_c \sim 140$.[20, 21] On the other hand the exact value of $\xi_L(\Gamma)$ at $\Gamma=2$ is only about twice the value it would have if the system was in a rigid triangular lattice and one averaged the fluctuations over translations and rotations.[17] The question then is how will $\xi_L(\Gamma)$ behave as Γ increases towards and beyond Γ_c? For comparison the Debye–Hückel length ξ_D, which $\xi_L(\Gamma)$ should approach as $\Gamma \to 0$,[8] decreases as $\Gamma^{-1/2}$. If this, or something resembling it, was also the behavior of $\xi_L(\Gamma)$ we would have a system with fluctuations much below that of the rigid lattice, which would be surprising indeed. It is this question which motivated the investigations described here.

The results of the simulations are given in Table I and Fig. 1. They show unambiguously that the decrease in $\hat{\kappa}(\Gamma)$ saturates for large Γ, approaching a value equal (certainly very close) to that of the triangular lattice. This is consistent with the intuition that there is a minimal value of the fluctuation per unit surface area and that this is achieved for a periodic

arrangement of points. On the other hand it was recently shown that randomly distributing the position of each particle in $\mathbb{Z}^d$ over a unit cell gives, for $d \gtrsim 350$, smaller fluctuations than the rigid lattice.[22] The question of the minimal value of $\hat{\kappa}$ in different dimensions is still open.

We remark here that we have tried, so far unsuccessfully, to come up with a scheme for generating translation invariant, mixing measures of particle distributions in $\mathbb{R}^d$ which would have surface growth of the variance. These would be measures on points that do not start with an equilibrium distribution of the OCP having a finite number of particles in a box of volume V and then take the thermodynamic limit of $V \nearrow \mathbb{R}^d$ in a appropriate way to obtain an infinite particle system with average density ρ.[12] It is known that the infinite volume measure of the OCP is not Gibbsian because the probability of large deviations in the number of particles in a region $\Lambda \in \mathbb{R}^d$ from its average value $\rho |\Lambda|$ behaves like $\exp[-C|\Lambda|^{\gamma}]$ with $\gamma > 1$.[23] One may wonder whether such behavior necessarily holds for all particle measures on $\mathbb{R}^d$ which have only surface growth of the variance.

III. MODEL AND COMPUTATIONAL DETAILS

In two dimensions the interaction between two particles of unit charge separated by a distance r is

$$v(r) = -\ln(r/L) \tag{7}$$

where L is an arbitrary unit length. A convenient unit of length, which will be used throughout the paper, is the radius of a disk containing one particle on the average, sometimes referred to as the "ion-disk radius," $a \sim (\pi\rho)^{-1/2}$. The reduced density is then $\rho = \pi^{-1}$ and a thermodynamic state is uniquely defined by the coupling constant Γ. The difficulties associated with computer simulations of this system due to the infinite range of the Coulomb interaction are well known. They are dealt with here, as in our previous work, by confining the particles to the surface of a sphere.[21]

For N particles of unit charge moving on the surface of a sphere of radius R with uniform background density of opposite charge the total potential energy is taken to be[21]

$$V_N = -\frac{1}{2}\sum_{i<j} \ln\left[\frac{2R^2}{L^2}(1-\mathbf{u}_i\cdot\mathbf{u}_j)\right] - \frac{N^2}{4}\left[1-\ln\frac{4R^2}{L^2}\right] \tag{8}$$

where $\mathbf{u}_i$ is a radial unit vector locating the position of particle i on the sphere surface. This corresponds to the distance between particles i and j

being measured along the chord joining the particles. In the thermodynamic limit ($N, R \to \infty$, $\rho = N/4\pi R^2$ constant) the energy differs from that of the planar system by a contribution of order $O(1/N)$.[21]

Most of our Monte Carlo simulations were performed with $N = 1024$ ions (i.e., $R = \sqrt{N}/2 = 16$); some at low and high couplings used $N = 2048$ ($R = 22.62$) to check the system size dependence. Charge fluctuations were calculated according to Eq. (6) which for the 2d OCP takes the form

$$\hat{\kappa} = (\langle N_\Lambda^2 \rangle - \langle N_\Lambda \rangle^2)/\mathscr{P}_\Lambda \tag{9}$$

where N_Λ is the number of particles in the domain Λ drawn on the surface of the sphere and $\mathscr{P}_\Lambda$ its perimeter. A convenient choice for Λ is a disk-like shape obtained by the intersection of the sphere with a cone of summit at the origin of the sphere and aperture θ. To check the independence of the results on surface shape additional computations were performed with a "rectangular" surface obtained by the intersection of the sphere with two planes parallel to and symmetric with respect to the equatorial plane of the simulation sphere and two parallel planes perpendicular to the equatorial plane.

IV. RESULTS

A. Charge Fluctuations

Results for $\hat{\kappa}$ in the range $\Gamma = 0.01 - 140$ covering the whole fluid domain are summarized in Table I and shown in Fig. 1. For each value of Γ, particle fluctuations were calculated for domains of different shape (disk- and rectangular-like) and different area. For instance, use of disk-like domains corresponding to $\theta = 81.1°$, 72°, 62,4° and 51,9°, and those symmetrical with respect to the center of the sphere, gave identical results (within statistical error) and were therefore averaged over. Results for rectangular domains (average over 8 domains) and disk-like domains were found to agree within statistical error (cf. Table I). When Γ is small $\hat{\kappa}$ decreases rapidly but then saturates at a value $\hat{\kappa} = 0.042 \pm 0.002$ near $\Gamma = 80$. For comparison, the corresponding value for $\hat{\kappa}$ for particles on a rigid triangular lattice is (in our units) 0.0404 while for the square lattice it is 0.0411.[17]

The excellent agreement of the simulation result with the exact value[18] $\hat{\kappa} = (2\pi\sqrt{\pi})^{-1} = 0.089793$ at $\Gamma = 2$ (cf. Table I) shows that both system size and domain dimensions are sufficiently large for reliable results

Table I. Particle Number Fluctuations $\hat{\kappa} = (\langle N_\Lambda^2 \rangle - \langle N_\Lambda \rangle^2)/\mathscr{P}$ in the 2D OCP as a Function of Γ.[a]

Γ	N	cycles	$\hat{\kappa}$	$\hat{\kappa}_{DH}$	Eqs. 12 and 14	ξ_L	ξ_D
0.01	1024	100000	1.035 ± 0.008				
0.01	2048	100000	1.080 ± 0.007	1.1254	1.1260	6.80	7.071
0.05	1024	100000	0.499 ± 0.001	0.5033	0.5047	3.135	3.162
0.10	1024	200000	0.356 ± 0.001	0.3559	0.3579	2.237	2.236
0.5	1024	300000	0.1638 ± 0.0006	0.1592	0.1638	1.029	1.00
1	1024	200000	0.1197 ± 0.0004	0.1125	0.1194	0.752	0.707
2_b	1024	100000	0.0897 ± 0.0004	0.0796	0.0898	0.564	0.50
4	1024	200000	0.0701 ± 0.0002	0.0563	0.0716	0.440	0.354
4^c	1024	100000	0.0696 ± 0.0003				
10	1024	100000	0.0548 ± 0.0005		0.0594	0.344	
20	1024	100000	0.0487 ± 0.0005			0.306	
40	1024	200000	0.0450 ± 0.0012			0.283	
40^c	1024	200000	0.0444 ± 0.0020				
60	1024	200000	0.0435 ± 0.0009			0.273	
80	1024	200000	0.0428 ± 0.0012			0.269	
80^c	1024	100000	0.0423 ± 0.0020				
100	1024	400000	0.0427 ± 0.0007				
100	2048	90000	0.0431 ± 0.0012			0.271	
120	1024	200000	0.0416 ± 0.0007			0.261	
140	1024	400000	0.0417 ± 0.0010				
140	2048	100000	0.0421 ± 0.0026		0.051	0.264	

[a] $\mathscr{P}$ is the perimeter of the surface of A. One cycle consists of trial translations of the N ions. $\hat{\kappa}_{DH} = 1/(2\pi\sqrt{2\Gamma})$, $\xi_D = 1/\sqrt{2\Gamma}$ and $\xi_L = 2\pi\hat{\kappa}$. Lengths are in units of $(\pi\rho)^{-1/2}$ and $e = 1$.
[b] The exact result is 0.089793.[19]
[c] Results for "rectangular" surfaces.

to be obtained. In fact system size dependence is observed only for $\Gamma \lesssim 0.01$. At these values of Γ both the 1024 and 2048 results differ from the Debye–Hückel limiting value $(2\pi\sqrt{2\Gamma})^{-1}$. This is presumably due to the fact that the correlation length ($\xi_L \sim \xi_D$) becomes comparable or exceeds the linear dimension of the domains.

From Eq. (6) it follows that $\hat{\kappa}$ can be expressed in terms of the usual pair correlation function $h(r) = g(r) - 1$

$$\hat{\kappa} = \frac{2}{\pi^2} \int_0^\infty dr\, r^2 h(r) \tag{10}$$

where r is in units of $a = (\pi\rho)^{-1/2}$.

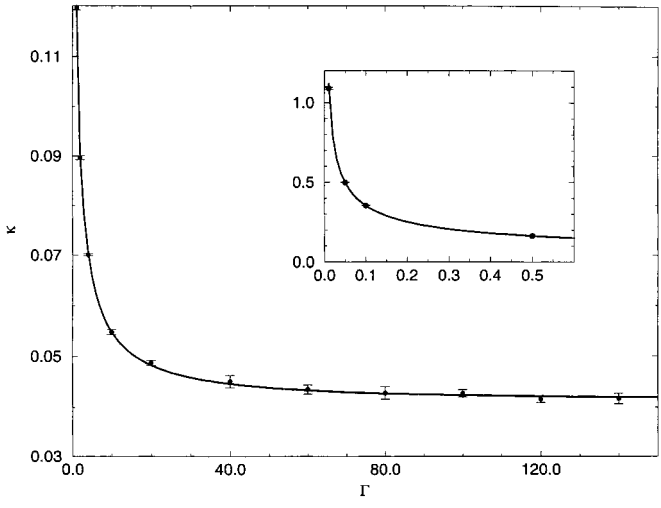

Fig. 1. Variation of $\hat{\kappa}$ with coupling Γ.

An extremely good representation of $\hat{\kappa}$ in the range $0 \leqslant \Gamma \leqslant 2$ can be obtained using in (10) an analytical approximation for the pair correlation function proposed in ref. 24

$$h(x) = -\frac{2}{\Gamma(\mu)} (x\sqrt{\mu})^{\mu} K_{\mu}(2x\sqrt{\mu}) \tag{11}$$

leading to the simple expression

$$\hat{\kappa} = \frac{1}{2\pi\sqrt{\pi}} \frac{1}{\sqrt{\mu}} \frac{\Gamma(\mu + 3/2)}{\Gamma(\mu + 1)} \tag{12}$$

where $\mu = \Gamma/(2-\Gamma)$ and $\Gamma(z)$ and $K_{\mu}(z)$ are the standard Gamma and Bessel functions, respectively. It is exact at $\Gamma = 0$ and $\Gamma = 2$ and reproduces the simulation data within statistical error. This is not so surprising since the zeroth, second and fourth moments of $h(x)$ (Eq. (11)) are exact (i.e., perfect screening, Stillinger–Lovett[2] and compressibility sum rules are satisfied) and therefore an accurate value for the first moment Eq. (11) can also be expected.

An analogous expression for $\Gamma \geqslant 2$

$$\hat{\kappa} = \frac{1}{2\pi\sqrt{\pi}} \frac{1}{\sqrt{\nu}} \frac{\Gamma(\nu)}{\Gamma(\nu - 1/2)} \tag{13}$$

based on Eq. (2.27) of ref. 24 ($\nu = \Gamma/(\Gamma - 2)$) is less satisfactory. Although accurate near $\Gamma = 2$ the subsequent decrease is too slow giving (by analytic continuation) a value $\hat{\kappa} = 1/2\pi^2 = 0.05066$ in the zero temperature limit.

An accurate fit reproducing all the data within statistical error is given by

$$\hat{\kappa} = \frac{0.11253954}{\Gamma^{1/2}} \frac{1 + a_1\Gamma^{1/2} + a_2\Gamma + a_3\Gamma^{3/2}}{1 + a_4\Gamma^{1/2} + \Gamma} \tag{14}$$

($a_1 = 3.9896$, $a_2 = 1.1211$, $a_3 = 0.38138$, $a_4 = 4.0934$). This form incorporates the exact Debye–Hückel limit at low Γ and saturates at high values of Γ.

B. Particle Distribution

As already alluded to in the Introduction it can be shown that under suitable clustering assumptions, which are fulfilled for the 2d OCP at small values of Γ (and at $\Gamma = 2$), the probability distribution of $Q_\Lambda/|\partial\Lambda|^{-1/2}$ is Gaussian in the limit $|\Lambda| \to \infty$, i.e., the probability distribution $P(N_\Lambda)$ of particles in a domain Λ has Gaussian behavior.[3, 4] To this end a histogram of particle number N_Λ was recorded during the simulation and fitted to a Gaussian with variance σ^2. The distributions $P(N^\Lambda)$, calculated in a disk-like domain with $\theta = 81.1°$, and its Gaussian fits are shown in Fig. 2 for $\Gamma = 0.5$, 2, 10 and 100. We find, for example, after taking an

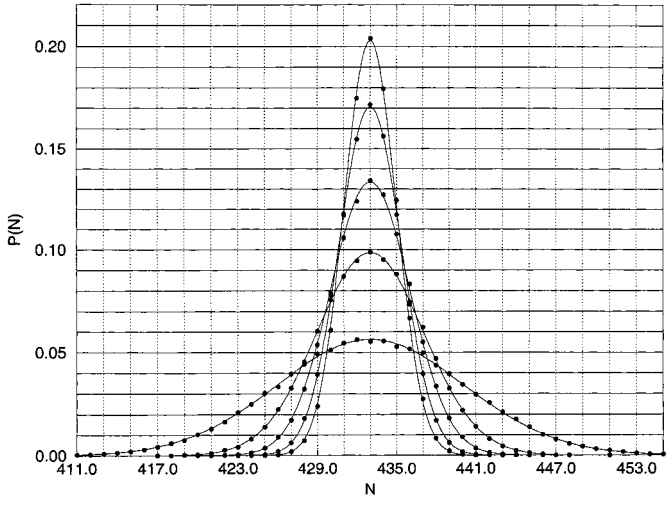

Fig. 2. Particle number distribution $P(N)$ for $\Gamma = 0.05$, 0.5, 2, 10, and 100 (from bottom to top). The filled circles are the Monte Carlo and the lines represent the Gaussian fits.

average over different domains A, that $\sigma^2/\mathscr{P} = 0.165$, 0.0901, 0.0551, 0.039 for $\Gamma = 0.5$, 2, 10, 100, respectively, in good agreement with the direct calculation (cf. Table I).

C. Screening Lengths

The Debye length ξ_D measures the range of correlations between pairs of charges in the limit of high temperature.[5, 6] For low values of Γ the asymptotic behavior of the pair correlation function exhibits exponential decay characterized by $\xi_D = 1/\sqrt{2\Gamma}$.[6, 10] This length can be compared with the length ξ_L typical of the spatial extension of charge fluctuations defined in (6). For the density fluctuations in our OCP this definition has the form

$$\hat{\kappa} = \frac{1}{2}\rho\xi_L = \frac{1}{2\pi}\xi_L \tag{15}$$

where ξ_L depends only on Γ and the factor $\frac{1}{2}$ is chosen so that ξ_L agrees with ξ_D when $\Gamma \to 0$.[8]

Table I shows that ξ_L is close to ξ_D for small values of Γ, the difference between ξ_L and ξ_D at $\Gamma = 2$ is $\sim 10\%$. For $\Gamma > 2$ an estimate of the correlation length of the pair correlation function is more problematic. As shown in Fig. 3 the pair correlation functions have a damped oscillatory

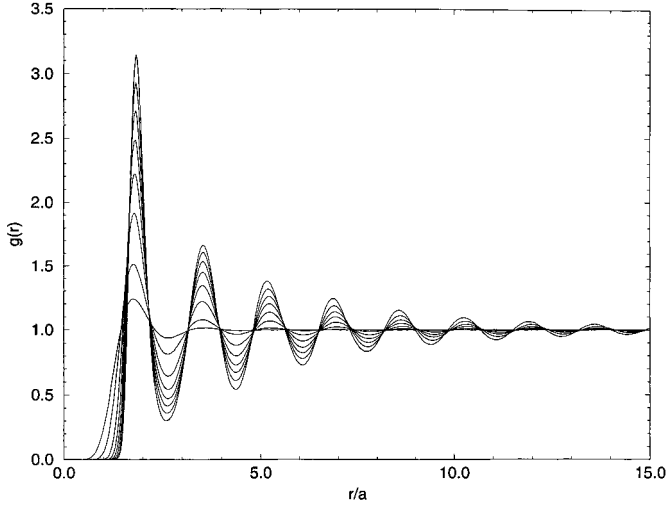

Fig. 3. Pair distribution function $g(r)$ for $\Gamma = 10$, 20, 40, 60, 80, 100, 120, and 140 (with increasing peak heights).

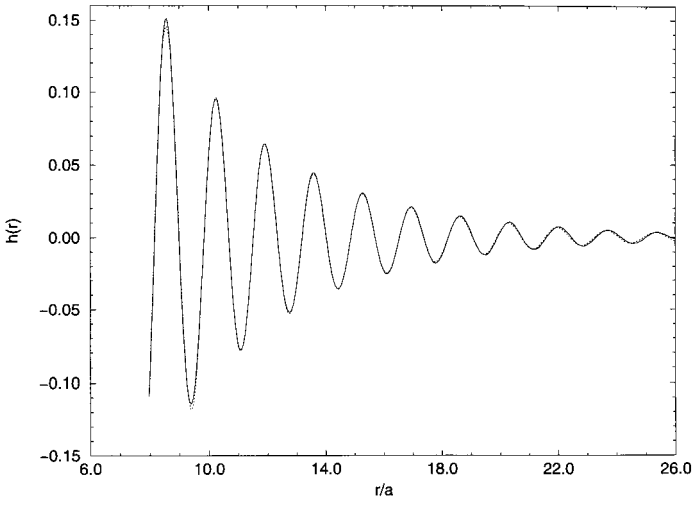

Fig. 4. Fit (dotted line) of the long range oscillations of the pair correlation function $h(r) = g(r) - 1$ for $\Gamma = 140$ by means of the functional form Eq. 16.

behavior at long range. These oscillations in $h(r)$ can be represented within statistical error by the expression

$$h(r) \sim Ae^{-\alpha r} \cos(\beta r - \gamma)/\sqrt{r}. \tag{16}$$

where A, α, γ are fitting parameters. An example of such a fit is shown in Fig. 4 for $\Gamma = 140$. Eq. (16) can be obtained by expressing $\tilde{h}(k)$, the Fourier transform of $h(r)$ in terms of the direct correlation function $\tilde{c}(k)$ according to $\tilde{h}(k) = \tilde{c}(k)/(1 - \rho\tilde{c}(k))$ and assuming that the long range behavior of $h(r)$ is driven by the poles in $(1 - \rho\tilde{c}(k))^{-1}$ closest to the real axis ("one" pole approximation studied, for instance, in ref. 25 for the 3d OCP). The parameters α and γ vary little with Γ. On the other hand α, which fixes the exponential damping of the correlations, decreases by a factor 3 between $\Gamma = 20$ and 140. (Table II). Moreover, between $\Gamma = 80$ and 140, α decreases

Table II. Parameter α of the Asymptotic Decay of the Pair Correlation Function $h(r) \approx Ae^{-\alpha r} \cos(\lambda r - \gamma)/\sqrt{r}$ (Where r is in Units of a, the Ion-Disk Radius)

Γ	20	40	60	80	100	120	140
α	0.769	0.549	0.404	0.326	0.278	0.225	0.194

by a factor 2 whereas $\hat{\kappa}$ is nearly constant. It thus appears that for large Γ the correlations lengths ξ_L and ξ_D are not related directly to the spatial decay of the envelope of $h(r)$ in the OCP, c.f. ref. 8. As mentioned above the exponential damping α is easily obtained from the "one pole approximation" but a more physical explanation of its origin has still to be found.

V. CONCLUDING REMARKS

Snapshots showing the arrangement of the ions on the sphere surface is given in Fig. 5 for $\Gamma = 2$ and 140. These show dramatically how the charges get more uniformly spaced as the temperature is lowered. On the other hand it would be hard to deduce from looking at the configuration at $\Gamma = 2$ (and even more so for smaller values of Γ) that the fluctuations were not normal: at $\Gamma = 2$, $h(r) = -e^{-\pi\rho r^2}$ so (6) is certainly valid.[19] We therefore have to go beyond visual inspection and this can be done more easily for a multiple component system than for the OCP.

As was noted in ref. 4 surface area growth of the variance in Coulomb systems can be interpreted as corresponding to the tendency of charged particles to form "bound" neutral entities. This is most readily visualized in a two component system of charges $\pm e$ and densities $\rho_1 = \rho_2 = \rho$. Suppose now that these charges could be paired somehow to make neutral dipoles with bond length D. Then the charge fluctuation in a region A could be interpreted as resulting from the boundary ∂A, cutting across some of the

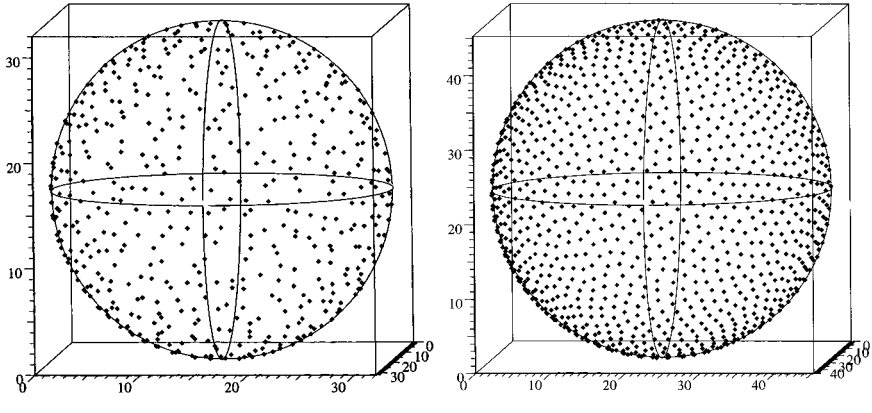

Fig. 5. Snapshots of particle configurations on the sphere surface at $\Gamma = 2$ (left graph) and 140 (right graph). Only the ions on the visible part of the surface are shown. (1024 particles were used for $\Gamma = 2$ and 2048 for $\Gamma = 140$.)

"bonds" between the charges connecting the ions in these molecules. We would then get by standard arguments (assuming sufficient independence between molecules far apart) a central limit theorem for the fluctuations in Q_Λ. The variance should then be as in (6), i.e., of order $Ce_2\rho\,|\partial\Lambda|$ with $D=\xi_L$. The constant C measures the effect of correlation between the orientations of the dipoles cut by $\partial\Lambda$ which is expected to be negative so C should be less than unity and decrease with Γ. The extension D might then be related to some screening length like ξ_D at high temperatures and to a "hard core" or other minimal distance length at low temperatures and low densities.[1, 8, 26] For the OCP $D\sim a$.

It is of course not necessary in the above analysis to insist on pairs of charges, or dipoles, being the basic neutral entity. We could equally have neutral quadrupoles made up of two positive and two negative charges or some hierarchical structure as in the analysis of the Kosterlitz–Thouless phase in $d=2$.[1, 26] What is necessary to make such an interpretation of charge fluctuations in Λ meaningful is that, in a typical configuration of the system, the neutral entities be spatially localized, i.e., not be greatly mixed up with other neutral entities. This is of course what happens in insulating materials, be they crystals, gases or liquids. Governed by quantum mechanics they consist of tightly bound neutral atoms or molecules. The picture in metals is similar in some ways to that of the OCP. Apparently enough of this picture remains true even in classical statistical mechanics of Coulomb systems, i.e., perfect screening, to give the correct behavior of the variance in Q_Λ.

ACKNOWLEDGMENTS

We would like to thank M. Aizenman, J. Beck, F. Cornu, S. Goldstein, B. Jancovici and Ph. Martin for helpful discussion and comments. Computing time on the CRAY C-98 was granted by the Institut de Développement et de Ressources en Informatique (IDRIS). The Laboratoire de Physique Théorique is Unité Mixte de Recherche N° 8627 du Centre National de la Recherche Scientifique. The research of J. L. L. was supported by AFOSR Grant F49620-98-1-0207, and NSF Grant DMR-9813268.

REFERENCES

1. G. Stell, *J. Stat. Phys.* **78**:197 (1995); G. Stell, in *New Approaches to Problems in Liquid State Theory*, C. Caccamo, J.-P. Hansen and G. Stell, eds. (Kluwer Academic Publishers, Dordrecht, 1999).
2. F. Stillinger and R. Lovett, *J. Chem. Phys.* **48**:3858 (1968).

3. H. van Beijeren and B. U. Felderhof, *Mol. Phys.* **38**:1179 (1979); Ch. Gruber, Ch. Lugrin, and Ph. A. Martin, *J. Stat. Phys.* **22**:193 (1980); Ph. A. Martin and T. Yalcin, *J. Stat. Phys.* **22**:435 (1980).
4. J. L. Lebowitz, *Phys. Rev. A* **72**:773 (1983); J. L. Lebowitz, in *Strongly Coupled Coulomb Systems*, G. J. Kalman, J. M. Rommel, and K. Blagoev, eds. (Plenum Press, New York and London, 1998).
5. Ph. A. Martin, *Rev. Mod. Phys.* **60**:1075 (1988).
6. D. Brydges and Ph. A. Martin, *J. Stat. Phys.* **96**:1163 (1999).
7. Ch. Gruber, J. L. Lebowitz, and Ph. A. Martin, *J. Chem. Phys.* **75**:994 (1981); L. Blum, Ch. Gruber, J. L. Lebowitz, and Ph. A. Martin, *Phys. Rev. Lett.* **48**:1767 (1982).
8. S. Bekiranov and M. E. Fisher, *Phys. Rev. Lett.* **81**:5836 (1998).
9. P. J. Forrester, B. Jancovici, and G. Téllez, *J. Stat. Phys.* **84**:359 (1996).
10. M. L. Mehta, *Random Matrices*, 2nd edition (Academic Press, 1990); O. Costin and J. L. Lebowitz, *Phys. Rev. Lett.* **75**:69 (1995).
11. D. C. Brydges and P. Federbush, *Commun. Math. Phys.* **73**:197 (1980); D. C. Brydges and P. Federbush, in *Rigorous Atomic and Molecular Physics*, G. Velo and A. Wightman, eds. (Erice Summer School, Plenum Press, Oxford), pp. 371–439.
12. J. Imbrie, *Commun. Math. Phys.* **87**:515 (1983).
13. J. L. Lebowitz and E. H. Lieb, *Phys. Rev. Lett.* **22**:631 (1969); E. H. Lieb and J. L. Lebowitz, *Adv. Math.* **9**:316 (1972); E. H. Lieb, *Rev. Mod. Phys.* **48**:553 (1976).
14. M. Baus and J. P. Hansen, *Phys. Rep.* **59**:1 (1980).
15. H. E. DeWitt and W. K. Slattery (and references there), in *Strongly Coupled Coulomb Systems*, G. J. Kalman, J. M. Rommel, and K. Blagoev, eds. (Plenum Press, New York and London, 1998).
16. J. Beck, *Acta Math.* **1**:159 (1987).
17. D. G. Kendall and R. A. Rankin, *Quart. J. Math.* (Oxford) **4**:178 (1953); see also P. Bleher, F. Dyson, and J. L. Lebowitz, *Phys. Rev. Lett.* **71**:3047 (1993).
18. M. Aizenman, S. Goldstein, and J. L. Lebowitz, in preparation; M. Aizenman and Ph. A. Martin, *Commun. Math. Phys.* **78**:99 (1980).
19. B. Jancovici, *Phys. Rev. Lett.* **46**:366 (1981).
20. S. W. de Leeuw and J. W. Perram, *Physica* **113A**:546 (1982).
21. J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen, *J. Stat. Phys.* **28**:325 (1982).
22. J. Beck, private communication.
23. B. Jancovici, J. L. Lebowitz, and G. Manificat, *J. Stat. Phys.* **72**:773 (1993).
24. J. Piasecki and D. Levesque, *J. Stat. Phys.* **47**:489 (1987).
25. R. J. F. Leote de Carvalho, R. Evans, and Y. Rosenfeld, *Phys. Rev. E* **59**:1435 (1999).
26. c.f. M. E. Fisher, *J. Stat. Phys.* **75**:1 (1994).
27. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **81**:527 (1981).